# CAPITAL AND OIL IN THE GLOBAL ECONOMY: OPTIMAL INVESTMENT AND FINANCIAL OPENNESS

*Georgy Trofimov*

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INSTITUTE FOR FINANCIAL STUDIES
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## ***ABSTRACT***

The global economy is energy-dependent because energy consumption is comprehensive, capital and energy are to a large extent complementary production factors, and international trade in the energy-carriers is highly specialized. A two-country model of this paper reflects these features and emphasizes a dynamic interdependence between capital accumulation and oil stock extension by oil-consuming and oil-producing economies. Financial openness under conditions of energy-dependence is necessary for optimal investment in the production factors separated by national border. The oil price plays in this model is endogenous and plays a key role in equalization of returns and the determination of optimal factor structure in the global economy. The model extension to the case of uncertainty in oil demand demonstrates that cross-country investments mitigate negatively correlated country-specific risks and allows households to build asset portfolios composed of non-diversifiable global risks which are the same for both economies.

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Georgiy Trofimov

## Introduction

Whether financial openness is important or redundant if nations trade in goods is a matter of a long-standing controversy. Cross-country risk sharing should provide insurance against country-specific risks and improve allocation of goods. But this improvement may be negligible in percentage of GDP if risks are generated by output shocks. Cole and Obstfeld (1990) demonstrated that in this case complete risk sharing may be delivered through terms-of-trade effects and there is no need in asset trade. This result explains the international diversification puzzle that financial portfolios are strongly home-biased in the world where trade in goods and assets is permitted and frictionless. On the contrary, Baxter and Jermann (1997) showed that returns to labor and domestic capital are highly correlated and hedging non-marketable human capital risk requires significant short positions in domestic marketable assets. Full diversification of country-specific risks would thus be possible under a very deep financial integration between industrial countries.

In this paper we do not concern the home-bias problem, but examine the role of financial openness for the case of an energy-dependent world economy with investment in production factors – capital and energy. This case markedly differs from the usual representation of the world economy by structurally similar industrial countries exchanging final goods which are imperfect substitutes. There is a high degree of specialization in the world energy production and a strong heterogeneity and interdependence between economies consuming and producing energy-carriers. On one hand, energy is an essential input for most economies with a high degree of complementarity with physical capital. Smith (2009) points at a very low oil demand elasticity which is equal to – 0.05 in the short run. On the other hand, global reserves of oil and natural gas are distributed very unevenly between developed and less developed economies and are clustered in the few oil and gas-rich countries.[1]

Because of the specialization in the production of final goods and energy, oil-consuming countries do not practice trade barriers against oil imports. Barriers are often imposed with regard to cross-border investment flows related to the world energy sector.[2]

[1] For instance, OPEC and the three former Soviet republics – Azerbaijan, Kazakhstan and Russia – produce together only 5.3 per cent of the world GDP but are endowed with 77.3 per cent of the total world oil reserves (according to International Energy Agency information).

[2] A tendency of financial protectionism strongly manifested before the global crisis of 2008-09, at the period of soaring world oil price. Some politicians in the developed countries advanced defensive measures against the sovereign oil funds engaged in strategic investment abroad that, supposedly, threatened national security. This tendency became even more prominent as the energy-dependence had got the status of a national security problem in the developed countries. Another side of the very same tendency is the strategy of resource nationalism exercised in some oil-exporting countries to maintain control over national resources.

Can one justify such a policy along the line of Cole-Obstfeld's reasoning that international asset trade is redundant in the oil-dependent world economy provided that trade in goods is free? We argue that in this case financial openness is important, apart from risk sharing, as a condition of optimal accumulation of the basic production factors – capital and oil reserves. To the extent that the economies specialize in trade and the world oil reserves are regionally clustered, these factors can be viewed as country-specific. The optimal factor accumulation path relies on a link between investment in capital and oil reserves which is absent under financial autarky. Trade only in goods is, hence, not enough for optimality of an equilibrium path if households invest only in home factors.

To formalize this argument we suggest a two-country model of trade and factor accumulation with oil as a production input complementary to physical capital. One of the countries is endowed with a capital stock and a final good manufacturing technology, while the other one – with an oil stock and an oil extraction technology. The economies are completely specialized in production and trade in which the final good is exchanged for oil (J7 and OPEC is a relevant case). Following the view of Adelman (1990)[3], we ignore the ultimate depletion of the world oil reserves which is indefinitely distant. The oil stock is assumed to be an inexhaustible factor playing in our model a dual role. It is a reservoir of oil and, at the same time, a production factor in oil extraction similar to capital in the final good manufacturing.[4] This property implies that the equilibrium oil price, which is endogenous in the model, is the sum of the marginal costs of oil stock maintaining and oil extraction. The former turns out to be constant, while the latter is increasing with *the global factor structure* – the world ratio of capital to oil stock – predetermined by investment decisions made previously by households in both countries.

Both capital and oil stock can be extended in any period through investment that predetermine the next period oil price which, in turn, defines the marginal expected returns on capital and oil stocks. The investment decisions are optimal if these returns are equalized across countries. This is ensured, without asset trade, by an implicit forward oil price which determines the next period optimal factor structure and investment. Under financial autarky investment decisions of households are independent, and the returns on capital and oil stock,

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Financial protectionism in both cases is motivated merely by political arguments like claims to prevent the threats of losing strategic controls.

[3] Adelman's (1990) critique of "the Hotelling theory" is based on the assumption of fixed stock of mineral resource: "There is no such thing. The total mineral in the earth is irrelevant non-binding constraint. If expected finding-development costs exceed the expected net revenues, investment dries up, and the industry disappears. Whatever is left in the ground is unknown, probably unknowable, but surely unimportant; a geological fact of no economic interest" (op. cit., p. 1).

[4] This kind of oil extraction technology has been used, for instance, by Devarajan and Fisher (1982) in a two-period model of natural resource extraction and reserve exploration under uncertainty.

generally, differ for a transitory equilibrium path. Opening of international bond market is sufficient for household investment to be optimal in the deterministic case.

The model is extended to the case of uncertainty in the oil-to-capital input coefficient causing fluctuations of inelastic oil demand. Under financial autarky the returns on capital and oil stock are negatively correlated, as well as the national incomes, indicating non-optimality of allocation of the world economy resources. Complete risk sharing occurs under equalization of the marginal rates of substitution across countries and states of nature implying equalization of the expected risk-adjusted returns on the production factors. The implicit forward oil price underlying risk sharing eliminates negatively correlated risks in the factor returns and ensures optimal investment.

This inference should be contrasted with Cole-Obstfeld's finding that asset trade may be redundant. Output shocks in their model cause fluctuations in the terms of trade that automatically pool risks, since a country's terms of trade are negatively correlated with growth in its export sector. For certain parameter choices the terms-of-trade responses alone provide perfect insurance against output shocks through complete risk sharing between countries.[5] On the contrary, the terms of trade in our model aggravate the effects of oil demand shocks on the world income distribution under financial autarky. The reason is that the marginal extraction cost (and the oil price) is increasing under a positive oil demand shock due to complementarity of capital and energy. Financial openness thus eliminates or mitigates the positive terms-of-trade effect.

We show that international investment is sufficient for a cross-country exchange of risks mitigating the shock-aggravating terms-of-trade effects. Opening of borders for direct and indirect foreign investment results in the equalization of risk-adjusted expected returns to the production factors. Though the risk sharing is incomplete and interstate marginal rates of substitution may vary between the countries, the global factor structure is near optimal for a small oil extraction rate (the ratio of oil extraction to oil stock). This is a consequence of the household asset portfolio structure under cross-country investment that brings about a positive correlation of risk premium in the factor returns. A risky *benchmark asset portfolio* corresponds to the global factor structure and is the same for both economies. Households are rewarded by the benchmark portfolio if the oil price falls, but this reward is larger on average

[5] Cole and Obstfeld used a Lucas-type (1982) two-country framework with investment in production factors and output shocks to show that for an important "knife-edge" case optimal allocation takes place under financial autarky. Heathcote and Perri (2009) demonstrated a somewhat similar result based on the Backus, Kehoe and Kydland (1995) model. They showed that changes in international relative prices provide insurance against country-specific shocks, and due to this portfolio choice does not play the key role in risk-sharing. Home bias in the Heathcote-Perri model arises because relative returns to domestic stocks move inversely with relative labor income in response to productivity shocks.

for a higher oil price. As a result, asset trade encourages households in both countries to invest in the production factors under upsurge of oil demand.

The next section presents the basic equilibrium model of the global economy under financial autarky and certainty. The deterministic model captures in a simple way the problem of country-specific factor accumulation and is presented for expositional purposes. Section 2 deals with a planner problem and conditions of optimal investment. Section 3 extends the basic model to uncertain oil demand and concerns the issue of risk sharing. Proofs of propositions are collected in appendix.

# 1. The basic model

In the basic model we consider a global economy that consists of two countries engaged in the bilateral exchange of goods for commodities. Country 1 is endowed with production capital and a manufacturing technology, and country 2 – with an oil stock and an oil extraction technology. The oil stock is renewable through investment in its maintenance and extension. Oil is absent in country 1 and manufacturing is absent in country 2.

Bilateral trade is essential for the global economy. Oil is exchanged for the final good on the competitive international markets under full specialization of the economies. All oil extracted in country 2 is exported to country 1 and used for the final good production as an intermediate energy input making the capital workable. The final good is used for consumption and investment in both countries and also as an intermediate input in oil extraction.

Each household in country 1 owns a final good-producing firm and in country 2 – an oil-extracting firm. Firms of each type are homogenous among households that make decisions on consumption and investment in the home production factors.

## 1.1. A two-country world

Countries differ in the size of population which is assumed constant. The number of households is $l_1$ in country 1 and $l_2$ in country 2, and the total world population is $l_1 + l_2 = 1$. Households are representative agents with homogenous preferences living indefinitely and maximizing the integral discounted utilities of consumption.

The country 1 household problem is to maximize

$$v_1 = \sum_{t=0}^{\infty} \beta^t u(c_{1t}) , \tag{1.1}$$

subject to a sequence of budget constraints per period

$$c_{1t} + i_{1t} = y_{1t}, \tag{1.2}$$

and a sequence of equations for capital accumulation:

$$k_{t+1} = (1-d)k_t + i_{1t}, \tag{1.3}$$

where $c_{1t}$ is the household consumption in period $t$, $u(c_{1t})$ is per-period utility, $\beta \in (0,1)$ is the discount factor, $y_{1t}$ is country 1 household income, $k_t$ is physical capital per capita, $d$ is the rate of capital depreciation, $i_{1t}$ is investment in capital.

A household is endowed with an initial capital stock $k_0$ and a technology for the final good production. The output per firm is a function of an oil-backed capital, $f_t = f(\hat{k}_t)$, which is a Leontieff function of physical capital and oil input:[6]

$$\hat{k}_t = \min(k_t, q_t / \theta). \tag{1.4}$$

Here $q_t$ is oil input into the final good production in period $t$, and $\theta$ is the parameter of oil input in relation to physical capital $k_t$. Production function $f(\hat{k}_t)$ is neoclassical and satisfying the Inada conditions.

The country 1 household is self-employed and supplies a unity of labor to its firm. The household income per period is equal to the final output, less the oil purchase from country 2:

$$y_{1t} = f(\hat{k}_t) - P_t q_t. \tag{1.5}$$

where $P_t$ is the oil price in units of the final good.

The household problem for country 2 is to maximize

$$v_2 = \sum_{t=1}^{\infty} \beta^t u(c_{2t}), \tag{1.6}$$

subject to the per-period budget constraints

$$c_{2t} + i_{2t} = y_{2t}, \tag{1.7}$$

and the oil stock equation:

$$s_{t+1} = s_t - x_t + i_{2t} / \delta, \tag{1.8}$$

where $c_{2t}$ is consumption, $s_t$ is the oil stock at the beginning of period $t$, $y_{2t}$ is the household income which is equal to the oil rent per field owned by the household, $x_t$ is oil extracted from the field, $i_{2t}$ is investment in the oil stock, $1/\delta$ is the marginal productivity

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[6] This is an extreme assumption adopted here for the sake of simplicity and to emphasize complementarity of energy and capital pointed out in the introduction. A more realistic approach allows for *ex ante* substitutability of energy and capital for newly installed equipment. For instance, Wei (2003) utilizes a putty-clay production technology with energy consumption and heterogeneous capital vintages.

of investment.[7] The latter provides maintenance of the oil stock equivalent to $x_t$ and its extension by the next period $s_{t+1} - s_t$. A household is endowed initially with oil stock $s_0$.

An oil field is identical to a competitive oil-producing firm. The oil rent per field is equal to the oil revenue less the extraction cost:

$$y_{2t} = P_t x_t - z_t, \tag{1.9}$$

where $z_t$ is the final good input in oil extraction. The oil extraction technology of the field is given by a homogenous of degree one, twice continuously differentiable production function of oil stock and intermediate input: $x_t = \Phi(s_t, z_t)$.[8] *An extraction rate* is the amount of oil extracted during one period in relation to the stock. It is monotonously increasing and strictly concave in the intermediate input intensity:

$$x_t / s_t = \phi(z_t / s_t) = \Phi(1, z_t / s_t), \tag{1.10}$$

satisfying the feasibility constraint $\phi(z_t / s_t) < 1$ and the Inada conditions: $\phi'(0) = \infty$, $\phi'(\infty) = 0$, and having the shape depicted in figure 1.[9] Under these assumptions the extraction is positive for any price and the stock cannot be exhausted in any period.

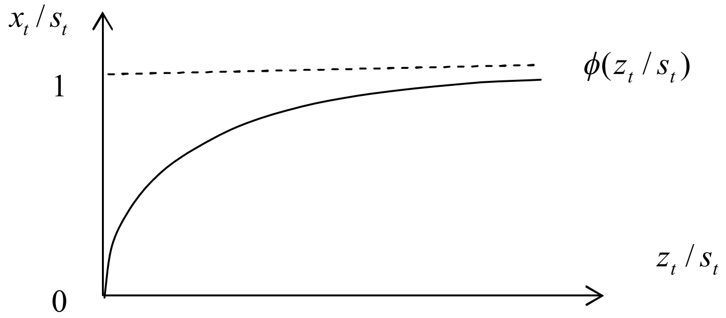


**Figure 1: The extraction rate function**

Equilibrium under financial autarky is a sequence of oil prices $P_t$ and plans of the final good production and oil extraction $f_t, q_t, x_t$, consumption-investment $c_{jt}, i_{jt}, j = 1,2$, and factor accumulation $k_t, s_t$ solving the households' problems and satisfying at any period the market-clearing condition:

$$Q_t = X_t, \tag{1.11}$$

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[7] The model would capture oil depletion if the productivity of investment in oil stock is assumed to be decreasing with cumulated volume of oil already extracted indicating scarcity: $\delta_t = \delta(\sum_{\tau=0}^{t} x_\tau)$, $\delta_t' > 0$.

[8] We ignore labor input which is insignificant in the oil industry. By the oil stock we mean proven reserves.

[9] An example of such function is $\phi(z_t / s_t) = \dfrac{(z_t / s_t)^\alpha}{1 + (z_t / s_t)^\alpha}$, $0 < \alpha < 1$.

where $Q_t = l_1 q_t$, $X_t = l_2 x_t$ is total demand and supply of oil. The market-clearing condition for the final good is fulfilled as identity:

$$F_t \equiv C_{1t} + C_{2t} + I_{1t} + I_{2t} + Z_t . \qquad (1.12)$$

where $F_t = l_1 f_t$ is final output, $C_{jt} = l_j c_{jt}, I_{jt} = l_j i_{jt}$ is aggregate consumption and investment in country *j*, $j = 1,2$, $Z_t = l_2 z_t$ is total intermediate input in oil extraction.[10]

## 1.2 Production and investment decisions

The decision on oil extraction is adopted in any period by a country 2 household to maximize the one-shot consumption utility[11] $u(c_{2t}) = u(P_t x_t - z(x_t, s_t) - \delta(s_{t+1} - s_t + x_t))$, where $z(x_t, s_t) = \phi^{-1}(x_t / s_t) s_t$ is the extraction cost function. The first-order condition for $x_t$ implies that the oil price is the sum of the marginal extraction cost and the marginal cost of oil stock maintenance compensating extraction in the current period:

$$P_t = p_t + \delta \qquad (1.13)$$

where $p_t = \partial z / \partial x_t$ is the marginal extraction cost. The equilibrium oil price (1.13) is the sum of the "static" term $p_t$ related to oil extraction in the current period and the "dynamic" term $\delta$ measuring the cost of oil stock restoring by the next period (or the alternative cost of oil extraction). The marginal extraction cost $p_t$ differs from the oil price $P_t$ by constant term $\delta$. For the sake of briefness the term $p_t$ will be called in what follows the oil price, as well as $P_t$, whenever it does not lead to confusion.[12]

The oil supply per field is $x_t = s_t \xi(p_t)$, where $\xi(p_t)$ is the equilibrium extraction rate as an increasing function of the oil price: $\xi(p_t) = \phi(\zeta(p_t^{-1}))$, $\zeta = \phi'^{-1} = z_t / s_t$ is the equilibrium intensity of intermediate input. The function $\xi(p_t)$ satisfies: $\xi(0) = 0$, $\lim_{p_t \to \infty} \xi(p_t) = 1$, $\lim_{p_t \to \infty} \xi'(p_t) = 0$. The oil extraction function $\phi(z_t / s_t)$ is assumed to have a sufficiently high curvature to guarantee that the equilibrium extraction rate $\xi(p_t)$ is strictly

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[10] The final output by country 1, $F_t$, differs from the global GDP $Y_{1t} + Y_{2t} = C_{1t} + C_{2t} + I_{1t} + I_{2t}$, where $Y_{1t} = l_1 y_{1t}$, $Y_{2t} = l_2 y_{2t}$.

[11] This results from combining the household budget constraint per period (1.7), the oil stock equation (1.8), and the oil rent equation (1.9).

[12] Under the model extension pointed out in footnote 7 and capturing oil depletion, the oil price equation (1.13) would transform as: $P_t = p_t + \delta_t + \delta_t' i_{2t} / \delta_t$. The second term is related to the oil stock maintenance and is increasing in time since $\delta_t' > 0$. The third term compensates for an increase of the cost of oil stock extension.

concave. This is the case for $\alpha < 1/2$ in the example of oil extraction technology given in footnote 9.

The oil rent per field is

$$y_{2t} = s_t(\delta\xi(p_t) + r(p_t)), \tag{1.14}$$

where $r(p_t) = p_t\xi(p_t) - \phi^{-1}(\xi(p_t))$ is monotonously increasing, strictly convex, and $r(0) = r'(0) = 0$. The marginal oil rent per barrel of oil in the ground is $\delta\xi(p_t) + r(p_t)$. It is the sum of the "dynamic" marginal rent $\delta\xi(p_t)$ compensating for an incremental decrease of the oil stock proportionally to the extraction rate, and of the "static" marginal rent $r(p_t)$ rewarding the owner of the oil field. The term $\phi^{-1}(\xi(p_t))$ is the extraction cost per barrel of oil in the ground as a function of the equilibrium extraction rate. The latter is linked with the static marginal oil rent as $\xi(p_t) = r'(p_t)$, according to Hotelling Lemma. Though $r'(p_t)$ converges to 1 as $p_t$ tends to infinity (since $\xi(p_t)$ converges to 1), the function $r(p_t)$ depicted in figure 2 has no asymptote line.[13]

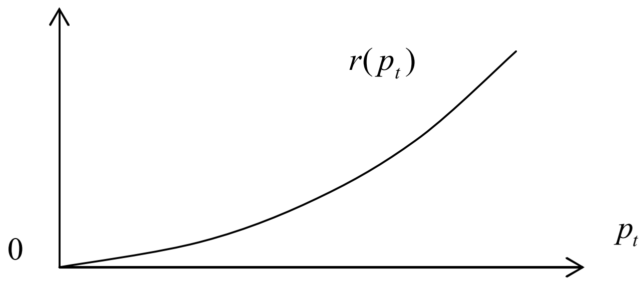


**Figure 2: The static marginal rent**

The final output per capita is $f(\hat{k}_t) = f(k_t)$, given that the production inputs are balanced: $k_t = q_t/\theta$. The oil demand by a country 1 household is $q_t = \theta k_t$. The oil market clears if $\theta k_t l_1 = s_t l_2 \xi(p_t)$, and the equilibrium oil price is

$$p_t = \varphi(\theta\kappa_t), \tag{1.15}$$

where $\varphi = \xi^{-1}$ is the inverse of the equilibrium extraction rate, and $\kappa_t = \lambda k_t / s_t$ is *the global factor structure* (the capital-to-oil stock ratio), $\lambda = l_1/l_2$ is the relative population size of

[13] $\lim_{p\to\infty}(r(p) - p) = \lim_{p\to\infty}[p(\xi(p) - 1) - \phi^{-1}(\xi(p)] = -\infty$ since $0 \le \lim_{p\to\infty} p(1 - \xi(p)) \le \infty$ and $\lim_{p\to\infty}\phi^{-1}(\xi(p)) = \infty$.

country 1. The function $\varphi$ is defined on the unit interval $[0,1)$, monotonously increasing, strictly convex, and $\varphi(0)=0$, $\varphi(1)=\infty$.

The country 1 household income is $y_{1t}=f(k_t)-P_t\theta k_t=f(k_t)-\delta\theta k_t-\theta p_t k_t$, according to (1.13). Taking into account the household budget constraint (1.2), the capital accumulation equation (1.3) is represented as

$$k_{t+1}+c_{1t}=f(k_t)+(1-D-\theta p_t)k_t \tag{1.16}$$

where $D=d+\delta\theta$. The marginal return on capital in period *t* is thus $R_t^k=1+f'(k_t)-D-\theta p_t$. It is equal to the gross marginal product of capital, $1+f'(k_t)$, less the physical depreciation rate $d$ and the oil purchase per unit of capital $\theta P_t=\theta(\delta+p_t)$. We call the term $D$ in (1.16) a *generalized* rate of capital depreciation to emphasize that the country 1 household income is the financial source for both the capital replacement in country 1 and oil stock maintenance in country 2. The oil stock "depreciates" at the rate of oil extraction $\xi(p_t)$ which is covered by term $\delta\theta$ of the generalized depreciation rate $D$.

Combining the oil stock equation (1.8) with the budget constraint (1.7), and the oil rent (1.14) yields $c_{2t}=y_{2t}-\delta(s_{t+1}-s_t+x_t)=\delta\xi(p_t)s_t+r(p_t)s_t-\delta(s_{t+1}-s_t+\xi(p_t)s_t)$ or

$$\hat{s}_{t+1}+c_{2t}=(1+r(p_t)/\delta)\hat{s}_t \tag{1.17}$$

where $\hat{s}_t=\delta s_t$ denotes the oil stock value in the final good unit.[14] The marginal return to this stock is $R_t^s=1+r(p_t)/\delta$. The returns $R_t^k$ and $R_t^s$ govern the intertemporal choices by households in both countries and do not, generally, coincide under financial autarky.

*Proposition 1. The equilibrium Euler equations are*

$$u'(c_{1t-1})=\beta R_t^k u'(c_{1t}), \tag{1.18}$$

$$u'(c_{2t-1})=\beta R_t^s u'(c_{2t}). \tag{1.19}$$

An equilibrium path of the global economy is given by the bundle ($p_t, k_t, s_t, c_{jt}, j=1,2$) satisfying the equations for the market-clearing oil price (1.15), the production factors accumulation (1.16), (1.17), and the intertemporal choice (1.18), (1.19). This path is determined by the initial endowments of capital and oil stocks and by the initial consumption choice satisfying the intertemporal budget constraints for each economy.[15]

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[14] In equilibrium the in-ground value of oil in the final good unit is equal to $\delta$ and is therefore constant in our model.

[15] Applying the dynamic programming would yield consumption functions as $c_{1t}=c_1(k_t,p_t)$, $c_{2t}=c_2(R_t^s s_t)$ and reduce a five-dimensional dynamic system for the global economy (1.15)-(1.19) to two difference equations on $k_t$ and $s_t$ (since $p_t$ is defined by the global factor structure).

## 1.3 The steady-state equilibrium

Consider a stationary equilibrium path along which all variables are constant in time. The Euler equations for this path imply that $\beta R_t^k = \beta R_t^s = 1$ or

$$f'(k) = \rho + D + \theta p, \tag{1.20}$$

$$r(p)/\delta = \rho, \tag{1.21}$$

where $\rho = \beta^{-1} - 1$ is the household discount rate. The time subscript is omitted in the notation of steady-state variables. Equation (1.20) defines a positive link between the marginal product of capital and the oil price. It extends the golden rule of capital accumulation, $f'(k_{gold}) = \rho + d$ . The higher is oil input coefficient $\theta$, the more significant is deviation from this rule, as figure 3 illustrates.

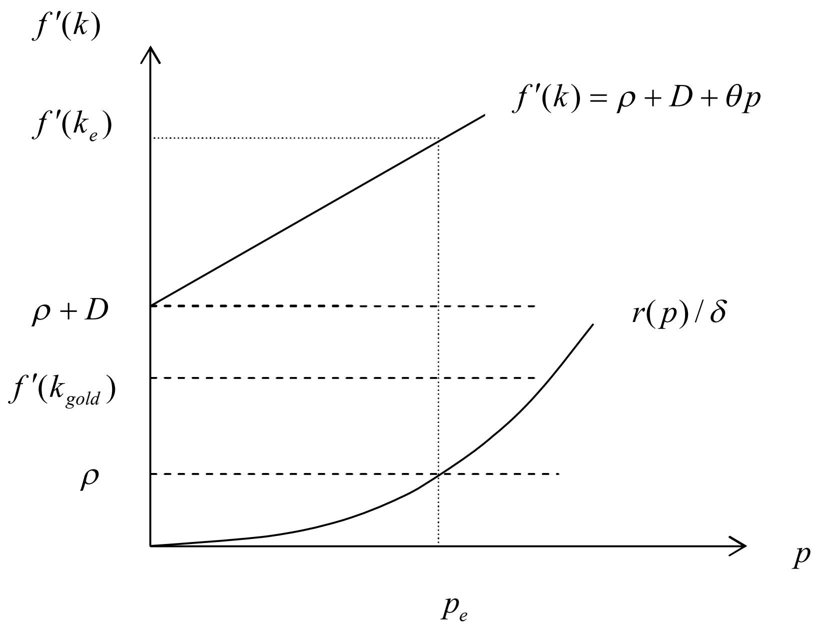


**Figure 3: The equilibrium steady-state**

The solution of (1.21) is the stationary oil price $p_e$ (the subscript *e* relates to the equilibrium steady-state). It is high if households are impatient ( $\rho$ is high) or investment in oil stocks are low-productive ( $\delta$ is high). Figure 3 shows the steady-state equilibrium defined by the pair $(p_e, k_e)$, where $f'(k_e) = \rho + D + \theta p_e$ . From the budget constraints (1.16), (1.17), the steady-state consumption is[16]

$$c_{1e} = f(k_e) - (D + \theta p_e)k_e \tag{1.22}$$

[16] The steady-state capital stock $k_e = f'^{-1}(\rho + D + \theta p_e)$ is decreasing in the oil price. The steady-state oil stock is calculated from the oil price equation (1.15) as $s_e = \theta\lambda k_e / \xi(p_e)$ . The steady-state investments are found from the factor accumulation equations (1.3), (1.8) as $i_{1e} = dk_e$, $i_{2e} = \delta x_e = \delta\theta k_e$ .

$$c_{2e} = \theta\lambda k_e r(p_e)/\xi(p_e) \quad (1.23)$$

Consumption in both countries positively relates to capital $k_e$ which is decreasing in the oil price $p_e$. Consumption in country 1 depends negatively on the oil price, while for country 2 this dependence is ambiguous. According to (1.23), all static rent is consumed fully by country 2, because maintenance of the steady-state oil stock is financed by country 1 through price $P_e$.

# 2. The global planner problem

Consider a planner maximizing the weighted integral utility $v^{\mu} = \mu l_1 v_1 + l_2 v_2$, where $\mu > 0$ is a relative weight placed on country 1. The individual utility is also weighted with the population size of countries $l_j, j = 1,2$. In any period the planner chooses investment, consumption, production, allocation of goods, and distribution of global income between the countries. The set of control variables include the oil price $P_t$ and a lump-sum transfer of the final good $T_t$ from country 1 to 2.

## 2.1. The optimal model

The problem is to maximize the integral weighted utility

$$v^{\mu} = \sum_{t=0}^{\infty} \beta^t \big(\mu l_1 u(c_{1t}) + l_2 u(c_{2t})\big), \quad (2.1)$$

subject to the condition of the final good distribution:

$$F_t = C_{1t} + C_{2t} + I_{1t} + I_{2t} + Z_t, \quad (2.2)$$

the resource constraint for oil:

$$\theta K_t \le X_t, \quad (2.3)$$

the budget constraint per period for country 1:

$$C_{1t} + I_{1t} = Y_{1t}, \quad (2.4)$$

and the equations for capital and oil stocks accumulation:

$$K_{t+1} = (1-d)K_t + I_{1t}, \quad (2.5)$$

$$S_{t+1} = S_t - X_t + I_{2t}/\delta . \quad (2.6)$$

where $K_t = l_1 k_t$ and $S_t = l_2 s_t$ is the total stock of capital and oil, respectively, $Y_{1t} = y_{1t} l_1 = F_t - P_t X_t - T_t$.

Equation (2.2) is fulfilled as identity (1.13) in the equilibrium model. The resource constraint (2.3) is implied from the Leontieff production function (1.5) represented on the aggregate level as $\hat{K}_t = \min(K_t, Q_t / \theta)$. From (2.2) and (2.4), the budget constraint for country 2 holds as identity: $C_{2t} + I_{2t} \equiv Y_{2t}$, where $Y_{2t} = y_{2t} l_2 = P_t X_t + T_t - Z_t$. Given the initial world stocks of capital and oil, $K_0$ and $S_0$, the planner chooses at any date the oil price $P_t$, the transfer $T_t$, the final good output $F_t$, the intermediate input $Z_t$, the oil extraction $X_t$, and the consumption-investment bundle $C_{jt}$, $I_{jt}$ ($j = 1,2$) solving the problem (2.1)-(2.6).

## 2.2 Optimal oil price and investment

The oil price and the transfer are the instruments of the world income distribution. It is shown in appendix that the optimal oil price ensures equalization of the weighted marginal household utilities:[17]

$$\mu u'(c_{1t}) = u'(c_{2t}) . \tag{2.7}$$

The higher is the relative weight of country 1, the larger is consumption per capita of this country relative to country 2. As is also shown in appendix, optimal oil price is the sum of the marginal costs of oil extraction and oil stock maintenance for any field, $P_t = p_t + \delta$. The optimal oil extraction and the oil rent per field coincide with the equilibrium ones: $x_t = s_t \xi(p_t)$, $y_{2t} = s_t(\delta\xi(p_t) + r(p_t))$. The equality of total oil supply and demand, $S_t \xi(p_t) = \theta K_t$, yields the oil price schedule the same as (1.15), $p_t = \varphi(\theta\kappa_t)$.

The equations for accumulation of factors are

$$K_{t+1} + C_{1t} = F(K_t) + (1 - D - \theta p_t) K_t - T_t , \tag{2.8}$$

$$\widehat{S}_{t+1} + C_{2t} = (1 + r(p_t)/\delta)\widehat{S}_t + T_t , \tag{2.9}$$

where $\widehat{S}_t = \delta S_t$ is the total oil stock measured in the final good units. In making investment decisions, the planner takes into account that the oil price $p_t$ depends on the global factor structure $\kappa_t$ predetermined by investment choice of the previous period. The right-hand sides of (2.8), (2.9) depend on this price, and the total effects of investment on consumption are captured by the marginal aggregate consumption rates with respect to the production factors, $\partial C_{jt} / \partial K_t$, $\partial C_{jt} / \partial \widehat{S}_t$, $j = 1,2$ .

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[17] The same first-order condition is obtained for the optimal transfer, but this does not mean redundancy of this control variable, as will be clear further below.

*Proposition 2. The marginal aggregate consumption rates for the planner's problem are*

$$\partial C_{1t} / \partial K_t = R_t^k - \Delta R_t^k, \;\; \partial C_{2t} / \partial K_t = \Delta R_t^k, \tag{2.10}$$

$$\partial C_{1t} / \partial \widehat{S}_t = \Delta R_t^s, \;\; \partial C_{2t} / \partial \widehat{S}_t = R_t^s - \Delta R_t^s, \tag{2.11}$$

*where* $\Delta R_t^k = \theta^2 \varphi'(\theta\kappa_t)\kappa_t$, $\Delta R_t^s = \theta\xi(p_t)\varphi'(\theta\kappa_t)\kappa_t / \delta$.

The marginal consumption rate of country 1 with respect to capital, $\partial C_{1t} / \partial K_t$, is equal to the return on capital $R_t^k$, less the marginal effect of capital growth $\Delta R_t^k$ causing an increase of the oil price and transferred to country 2. Similarly, the marginal consumption rate $\partial C_{2t} / \partial \widehat{S}_t$ is equal to the oil stock return $R_t^s$, less the marginal effect of the oil stock extension $\Delta R_t^s$ causing a decrease of the oil price and transferred to country 1. The planner has to internalize these distribution effects, ignored by households. But being captured through the optimal oil price ensuring (2.7), these effects can be neglected in optimal investment decision, as the next proposition demonstrates.

*Proposition 3. The optimal consumption-investment path satisfies:*

$$u'(c_{1t-1}) = \beta R_t^k u'(c_{1t}), \tag{2.12}$$

$$u'(c_{2t-1}) = \beta R_t^s u'(c_{2t}). \tag{2.13}$$

Though these equations coincide with (1.19), (1.20), the optimal and equilibrium paths are, generally, not identical because the returns $R_t^k$ and $R_t^s$ are determined differently in the optimal and equilibrium models. The optimal oil price condition (2.7) implies that the marginal utilities ratio $u'(c_{1t}) / u'(c_{2t})$ is constant in time and, hence, the marginal returns on capital and the oil stock are equalized at any period, $R_t^k = R_t^s$.

This requirement imposes a condition on the optimal oil price:

$$r(p_t) / \delta + \theta p_t = f'(k_t) - D, \tag{2.14}$$

Since the left-hand side of this equation is monotonously increasing in $p_t$, it has one positive root $p_t^*$, given that $f'(k_t) > D$ (the asterisk refers to optimal solution). From (2.14), the optimal oil price is a decreasing function of capital: $p_t^* = p(k_t)$, $p'(k_t) < 0$, and it is also decreasing in $\theta$. The reason for this is that any increase of oil demand leads to a decrease of $R_t^k$ and an increase of $R_t^s$. To cancel these effects out, the optimal oil price decreases exerting a "stabilizing" comparative static effect on the returns.

Inserting $p(k_t)$ into the country 1 budget constraint (2.8) written as $c_{1t} = f(k_t) + (1 - D - \theta p(k_t))k_t - \tau_{1t} - k_{t+1}$, where $\tau_{1t} = T_t / l_1$, and inserting the optimal return to capital $R^k(k_t) = 1 + f'(k_t) - D - \theta p(k_t)$ into the Euler equation (2.12) yields a dynamic system for $c_{1t}, k_t$. It defines the optimal path for country 1, given the optimal transfer path $T_t$, irrespective of country 2 path. The latter is predetermined by the capital path in the following way.

The returns-equalizing oil price $p(k_t)$ must coincide with the market-clearing oil price: $p_t = \varphi(\theta\kappa_t)$. This is fulfilled if $\theta\kappa_t = \varphi^{-1}(p(k_t)) = \xi(p(k_t))$ implying that the optimal oil stock is

$$s(k_t) = \theta\lambda k_t / \xi(p(k_t)). \tag{2.15}$$

This stock is an increasing and convex function of capital since $\xi'(p_t) > 0$, $p'(k_t) < 0$. Due to (2.15), the volume of investment required to ensure the stock $\hat{s}(k_{t+1})$ is $i_{2t}^* = \hat{s}(k_{t+1}) - \hat{s}(k_t) + \delta\lambda\theta k_t$. Then consumption in country 2 should satisfy the budget constraint (2.9) represented as $c_{2t} = (1 + r(p(k_t))/\delta)\hat{s}(k_t) + \tau_{2t} - \hat{s}(k_{t+1})$, $\tau_{2t} = T_t / l_2$, and the Euler equation (2.13) with $p_t = p(k_t)$. These two equations are compatible in any period due to the choice of transfer.

Given the initial capital stock, $k_0$, the optimal path of country 1 is selected through the choice of $c_{10}^*$ satisfying the intertemporal budget constraint for this country (obtained by iterating (2.8) over time):

$$\sum_{t=0}^{\infty} \left(\Pi_{i=0}^{t} R^k(k_i)\right)^{-1} (c_{1t} + i_{1t} + \tau_{1t} - y_{1t}) = 0. \tag{2.16}$$

The initial consumption for country 2 is found from (2.7) as $c_{20}^* = u'^{-1}\left(\mu^{-1} u'(c_{10}^*)\right)$ or $c_{20}^* = \mu^{-1/\sigma} c_{10}^*$ for the case of isoelastic utility $u(c) = (c^{1-\sigma} - 1)/(\sigma - 1)$, $\sigma > 0$. This relationship should be valid for the initial oil stock $s_0$ and any relative utility weight $\mu$. The latter is matched with the time-averaged expected transfer $\bar{T}$ derived from the intertemporal budget constraint for country 2 similar to (2.16): $\bar{T} = \sum_{t=0}^{\infty} \vartheta_t T_t$, where $\vartheta_t = \sum_{\tau=0}^{t} \left(\Pi_{i=0}^{\tau} R^k(k_i)\right)^{-1} / \sum_{\tau=0}^{\infty} \left(\Pi_{i=0}^{\tau} R^k(k_i)\right)^{-1}$ is the time weight of period-$t$ transfer.

## 2.3 The steady state

Along the stationary optimal path the Euler equation (2.12) for country 1 transforms into $\beta R^k(k) = 1$ or $f'(k) = \rho + D + \theta p(k)$. The stationary optimal oil price coincides with the stationary equilibrium oil price,[18] $p(k) = p_e$ implying that the optimal capital stock is the same as in the steady-state equilibrium, $k^* = k_e$. The optimal oil stock is also the same, $s^* = s_e = \theta\lambda k_e / \xi(p_e)$, as well as investments in these stocks: $i_1^* = dk_e$, $i_2^* = \delta\theta k_e$.

Optimal consumption, however, differs from the equilibrium one due to the transfer:

$$c_1^* = f(k_e) - (D + \theta p_e)k_e - \tau_1^* \tag{2.17}$$

$$c_2^* = \theta\lambda k_e r(p_e) / \xi(p_e) + \tau_2^* \tag{2.18}$$

where $\tau_j^* = T^* / l_j$ is the stationary optimal transfer per capita, $j = 1,2$. It is calculated for the isoelastic utility from the marginal utility equalization (2.7) implying $c_1^* = \mu^{1/\sigma} c_2^*$ for isoelastic utility. Combining this with (2.17), (2.18) and rearranging terms yields $T^* = \dfrac{f(k_e) - \left(D + \theta(p_e + \mu^{1/\sigma}\lambda r(p_e)/\xi(p_e))\right)k_e}{l_1^{-1} + \mu^{1/\sigma} l_2^{-1}}$. A higher utility weight of country 1 implies a lower transfer to country 2.

The equilibrium and optimal stationary paths thus coincide for all variables but consumption which is shifted due to the transfer. The equilibrium steady-state path is optimal for an utility weight corresponding to zero transfer, $T = 0$.

## 2.4 The international bond market

Since the Euler equations coincide for the equilibrium and optimal models, the planner can rely on individual choices of consumption and investment, provided that the factor returns are equalized.[19] The financial autarky does not ensure it (except for the stationary path). Consider the simplest case of financial openness. Let households trade in one-period bonds with risk-free return $R_t$, and $b_{jt}$ denote the foreign bond holding by a country *j* household. The household budget constraints are rewritten as $k_{t+1} + b_{1t+1} + c_{1t+1} = f(k_t) + (1 - D - \theta p_t)k_t + R_t b_{1t}$ for country 1, and $\hat{s}_{t+1} + b_{2t+1} + c_{2t} = R_t^s \hat{s}_t + R_t b_{2t}$ for country 2.

The bond market is in equilibrium, $\lambda b_{1t} + b_{2t} = 0$, only if the returns on bonds and production factors are equalized: $R_t = R_t^k = R_t^s$. The second equality is fulfilled for the optimal price $p(k_t)$ determined in period $t-1$ simultaneously with $R_t^k$ and $R_t^s$. One can

[18] Inserting $f'(k)$ into the optimal oil price equation (2.14), yields $r(p(k))/\delta + \theta p(k) - (f'(k) - D) = r(p(k))/\delta + \theta p(k) - (\rho + \theta p(k)) = 0$ or $r(p(k))/\delta = \rho$ which is equivalent to the equilibrium steady-state Euler equation (1.21).

[19] The effect of investment on the oil price and the factor returns is neutralized in the optimal model by the redistribution effect of the marginal utilities equalization. Individuals make investment decisions as if they capture these effects by neglecting both.

interpret it as a forward oil price $p_t^f$ contracted in period $t-1$ and eliminating arbitrage opportunities. It coincides with the spot market-clearing price in period $t$, $p_t^f = p_t$. Households in country 2 infer from this that the oil stock is linked with capital according to (2.15). They make consumption-investment decisions based on the capital-linked investment schedule $i_{2t-1}^*$ and take into account the budget constraint (1.7) and the Euler equation (1.19). These conditions are compatible due to the bond issues or purchases $b_{2t} - R_t b_{2t-1}$ similar to transfer $\tau_{2t-1}$ in the planner's model.

Consequently, international trade in bonds ensures optimality of equilibrium path. The initial debt per capita $b_{j0}$ corresponds to utility weight $\mu$ in the same way as the time-average transfer $\overline{T}$ corresponds to this weight in the planner's problem.

# 3. A model with uncertainty

The basic model is extended in this section to consider risk sharing between the economies and investment under uncertainty. It is assumed that the oil demand parameter $\theta_t$ is stochastic and driven by a Markov chain with the finite number of states $N$. The realizations $\theta_t = \theta^i$ are ordered as $\theta^1 < \theta^2 < ... < \theta^N$, and the transition probability is $\pi^{ig} = \Pr(\theta_t = \theta^g \middle| \theta_{t-1} = \theta^i)$, $\sum_{g=1}^{n} \pi^{ig} = 1$.

## 3.1 Equilibrium under financial autarky

Households maximize the expected value of consumption, $v_j = E_0 \sum_{t=0}^{\infty} \beta^t u(c_{jt})$, $j = 1,2$, subject to the budget constraints and the factor accumulation equations that are the same as in the basic equilibrium model. Production and trade take place in each period $t$ after realization of $\theta_t$, but investments in production factors are decided in period $t-1$. Though the production factors $k_t$ and $s_t$, as well as the final output $f(k_t)$, are known with certainty by the beginning of period $t$, the global income distribution is uncertain because the equilibrium oil price $P_t(\theta_t)$ is unknown.

The static equilibrium conditions derived above for the deterministic model are valid for any $\theta_t$ observed in period $t$: the equilibrium oil price is $P_t(\theta_t) = \delta + p_t(\theta_t)$ where $p_t(\theta_t) = \varphi(\theta_t \kappa_t)$ is the marginal extraction cost term as the function of the oil input coefficient times the factor structure. The household incomes are $y_{1t}(\theta_t) = f(k_t) - dk_t - (\delta + p_t(\theta_t))\theta_t k_t$, $y_{2t}(\theta_t) = s_t(\delta\xi(p_t(\theta_t)) + r(p_t(\theta_t)))$. The capital return $R_t^k(\theta_t) = 1 + f'(k_t) - D(\theta_t) - p_t(\theta_t)\theta_t$, $D(\theta_t) = d + \delta\theta_t$, is negatively correlated with the oil demand shocks $\theta_t$, while the oil stock return $R_t^s(\theta_t) = 1 + r(p_t(\theta_t))/\delta$ is positively correlated. The stochastic equilibrium Euler equations are

$$u'(c_{1t-1}) = \beta E_{t-1} R_t^k u'(c_{1t}), \tag{3.1}$$

$$u'(c_{2t-1}) = \beta E_{t-1} R_t^s u'(c_{2t}). \tag{3.2}$$

where $E_{t-1} R_t^l u'(c_{jt}) = \sum_{g=1}^{N} R_t^l(\theta^g) u'(c_{jt}(\theta^g))\pi^{ig}$, given that $\theta_{t-1} = \theta^i$, is the conditional expectation built on information of date $t-1$ and the Markov transition probability ($l = k, j = 1$ or $l = s, j = 2$).

Under stationary stochastic equilibrium, consumption in country $j = 1,2$ and production factors are contingent on states at any date $t$: $c_{je}(\theta^i) = c_{je}^i$, $k_e(\theta^i) = k_e^i$, $s_e(\theta^i) = s_e^i$ if $\theta_t = \theta^i$.[20] This equilibrium is in general inefficient since the marginal rates of substitution $u'(c_{je}^g)/u'(c_{je}^i)$ are not equalized across states unless the risks are traded between the countries.

## 3.2 The planner's problem

The planner maximizes the expected weighted integral utility $v^\mu = E_0 \sum_{t=0}^{\infty} \beta^t \left(\mu l_1 u(c_{1t}) + l_2 u(c_{2t})\right)$ subject to the equations for global income distribution (2.2), (2.4)-(2.6), and the resource constraint for oil (2.3) written as $\theta_t K_t \le X_t$. The global economy is subject to aggregate uncertainty since the oil demand shock $\theta_t$ affects input decisions that cannot be eliminated through allocation of goods in period $t$.

The planner uses the same policy tools as in the deterministic model including the oil price $P_t(\theta_t)$ and the lump-sum transfer $T_t(\theta_t)$ from country 1 to 2 that are chosen after $\theta_t$ has been observed. Weighted marginal utilities are equalized at any state:

$$\mu u'(c_{1t}(\theta^i)) = u'(c_{2t}(\theta^i)), \tag{3.3}$$

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[20] The oil-augmented golden rule of capital accumulation (1.20) and the marginal oil rent equation (1.21) are generalized as $f'(k_e^i) = \rho + d + \sum_{g=1}^{N} \theta^g (\delta + \varphi(\theta^g \kappa_e^i))\pi^{ig} \chi_1^{ig}$ and $\sum_{g=1}^{N} r(\varphi(\theta^g \kappa_e^i))\pi^{ig} \chi_2^{ig} = \delta\rho$, respectively, where $\kappa_e^i = \lambda k_e^i / s_e^i$, $\chi_j^{ig} = u'(c_{je}^g)/u'(c_{je}^i)$.

$i = 1,...,N$. Once the uncertainty about $\theta_t$ is resolved, $\theta_t = \theta^i$, the static optimal plans of production and trade in goods are calculated in the same way as in the deterministic model. As above, the effect of investment on the next-period oil price can be neglected under (3.3). Let $\widetilde{E}_{jt-1}$ denote a conditional expectation for country *j* defined on *the risk-adjusted* probability distribution $\widetilde{\pi}^{ig}_{jt} = \pi^{ig} u'(c_{jt}(\theta^g)) / E_{t-1} u'(c_{jt})$, given that $\theta_{t-1} = \theta^i$.

*Proposition 4. (i) The optimal path satisfies the stochastic Euler equations*

$$u'(c_{jt-1}) = \beta E_{t-1} R^l_t u'(c_{jt}), \tag{3.4}$$

*where $l = k, j = 1$ or $l = s, j = 2$. For this path: ii) the risk-adjusted transition probabilities are the same for both countries:*

$$\widetilde{\pi}^{ig}_{1t} = \widetilde{\pi}^{ig}_{2t} \equiv \widetilde{\pi}^{ig}_t$$

*for $i, g = 1,...,N$, and iii) the risk-adjusted expected returns are equalized:*

$$\widetilde{E}_{t-1} R^k_t = \widetilde{E}_{t-1} R^s_t. \tag{3.5}$$

The country subscript is omitted in the notation of conditional risk-adjusted expectation $\widetilde{E}_{t-1}$ because it is the same for both countries due to the equalization of risk-adjusted probabilities. The optimal and equilibrium investment plans coincide due to (3.4) and (3.5). The latter equation is represented as $\widetilde{E}_{t-1}[r(p^*_t(\theta_t)) / \delta + \theta_t p^*_t(\theta_t) + \theta_t \delta] = f'(k_t) - d$. Since the optimal price must ensure market clearing, $p^*_t(\theta_t) = \varphi(\theta_t \kappa_t)$, we have:

$$\widetilde{E}_{t-1}[r(\varphi(\theta_t \kappa_t)) / \delta + \theta_t \varphi(\theta_t \kappa_t) + \theta_t \delta] = f'(k_t) - d. \tag{3.6}$$

This is an equation for optimal factor structure $\kappa_t$ extending (2.14). Its left-hand side is increasing in $\kappa_t$, implying the unique solution $\kappa^*_t$. Consider a second-order approximation for $\kappa^*_t$ by assuming that the extraction rate $\theta_t \kappa_t$ is small[21]. Suppose that the oil extraction function $\phi(z_t / s_t)$ is of sufficiently high curvature near zero to guarantee that $\xi'(0) = \infty$, $-\xi''(0) / \xi'(0)^3 = \varphi''(0) = 0$ (the latter condition is fulfilled for our reference example in footnote 9 for $\alpha < 1/3$).

*Proposition 5. The squared optimal factor structure is approximately equal to*

$$\kappa^{*2}_t \approx 2\delta \frac{f'(k_t) - d - \delta \widetilde{E}_{t-1} \theta_t}{\widetilde{E}_{t-1} \theta^2_t}. \tag{3.7}$$

[21] The assumption of small extraction rate fits the stylized facts about the global oil industry. Smith (2009, p. 153) points out that OPEC's installed production facilities are sufficient to extract 1.5 percent of its proved reserves per year, while non-OPEC producers have installed facilities sufficient to extract 5.6 of their proved reserves each year.

The nominator in the right-hand side of (3.7) is the marginal product of capital in period $t$, net off the expected generalized depreciation rate. The denominator is the second-order moment of $\theta_t$ built on the risk-adjusted transition probabilities $\tilde{\pi}_t^{ig}$. According to (3.7), the expected return on capital is equal to the risk premium calculated as $\kappa_t^{*2}\tilde{E}_{t-1}\theta_t^2 / 2\delta$ and rewarding households in both countries for the adoption of the risky factor structure.

The optimal oil stock is equal to $s_t^* = \lambda k_t / \kappa_t^*$ or, from (3.7) [22]

$$s_t^* = \lambda k_t \left( \frac{\tilde{E}_{t-1}\theta_t^2}{2\delta(f'(k_t) - d - \delta\tilde{E}_{t-1}\theta_t)} \right)^{1/2}. \tag{3.8}$$

It is increasing and convex in capital entering explicitly the right-hand side of (3.8), and is decreasing in the conditional moments $\tilde{E}_{t-1}\theta_t$ and $\tilde{E}_{t-1}\theta_t^2$. By the Markov property of $\theta_t$, all information about this parameter available in period $t-1$ is contained in $\theta_{t-1}$. One can assume that the structure of the transition probability matrix is close enough to diagonal to guarantee a strong positive autocorrelation for $\theta_t$ sufficient for both conditional moments $\tilde{E}_{t-1}\theta_t$ and $\tilde{E}_{t-1}\theta_t^2$ to be increasing in $\theta_{t-1}$. Then, from (3.8), $s_t^*$ is increasing in $\theta_{t-1}$.

This property implies a qualitative rule for optimal investment policy requiring that $\kappa_t^*$ should decrease in response to a positive oil demand shock. Optimal oil stock investment exerts, according to (3.8), a stabilizing effect on the optimal oil price. Realization of high $\theta_{t-1}$ causes an upward movement of this price in period $t-1$ but, according to (3.7), there is a downward effect in period $t$ on $\varphi(\theta_t \kappa_t^*)$ through a decrease of the optimal factor structure.

## 3.3 State-contingent securities

Consider a market for one-period Arrow securities providing mutual insurance for countries. In states with low $\theta^g$ country 1 is an insurer, while in states with high $\theta^g$ it is insured by securities purchased from country 2. A security issued in period $t-1$ is a promise to pay a dollar in each state $g = 1,...,N$ in period $t$. Let $a_{jt}^g$ denote state-$g$ security purchased in period $t-1$ by a household from country $j = 1,2$ at price $\psi_{t-1}^{ig}$ under state $\theta_{t-1} = \theta^i$. The household budget constraints are

$$k_t + a_{1t} + c_{1t-1} = f(k_{t-1}) + (1 - D(\theta_{t-1}) - \theta_{t-1} p_{t-1}(\theta_{t-1}))k_{t-1} + a_{1t-1}^i, \tag{3.9}$$

$$\hat{s}_t + a_{2t} + c_{2t-1} = R_{t-1}^s(\theta_{t-1})\hat{s}_{t-1} + a_{2t-1}^i, \tag{3.10}$$

[22] As above, consumption plans of the countries are compatible due to the choice of transfers $\tau_{2t-1}(\theta_{t-1})$.

where $a_{jt} = \sum_{g=1}^{N} \psi_t^{ig} a_{jt}^g$ is the total purchase of state-contingent securities by the household. The markets for these securities clear if $l_1 a_{1t}^g + l_2 a_{2t}^g = 0$ for $g = 1,...,N$.

*Proposition 6. Trade in state-contingent securities yields: i) equalization of the risk-adjusted transition probabilities across countries for any state:*

$$\tilde{\pi}_{1t}^{ig} = \tilde{\pi}_{2t}^{ig} \equiv \tilde{\pi}_t^{ig}, \tag{3.11}$$

*ii) equalization of the risk-adjusted returns:*

$$\tilde{E}_{t-1} R_t^k = \tilde{E}_{t-1} R_t^s = R_t, \tag{3.12}$$

*where* $R_t = 1 / \sum_{g=1}^{N} \psi_{t-1}^{ig}$ *is the risk-free rate of return, and* $\psi_{t-1}^{ig} = \tilde{\pi}_t^{ig} / R_t$.

The risk-free return in (3.12) is yielded by a uniformly weighted portfolio of securities. The factor returns are equalized for $\kappa_t = \kappa_t^*$ due to identity of the first-order conditions for investment (3.5) and (3.12). The market-clearing oil price is optimal in any state $\theta_t = \theta^g$ since $p_t^*(\theta^g) = \varphi(\theta^g \kappa_t^*)$, and, as a result, households make optimal investment decisions. The implicit forward oil price is the mean of state-contingent prices:[23] $p_t^f = \tilde{E}_{t-1}\theta_t p_t^*(\theta_t) / \tilde{E}_{t-1}\theta_t$ . It ensures equalization of risk-adjusted expected returns similarly to (2.14): $\tilde{E}_{t-1} r(p_t^*(\theta_t)) / \delta + p_t^f \tilde{E}_{t-1}\theta_t = f'(k_t) - \tilde{E}_{t-1} D(\theta_t)$.

## 3.4 Cross-country investments and the global asset portfolio

The assumption of complete markets is essentially relaxed if we introduce cross-country investment in the production factors. Suppose that households in both countries make direct and indirect investments in capital and oil stocks and also trade in one-period risk-free bonds. Indirect investment is an amount of capital or oil stock acquired by a household in a perfect international market for production factors. There are no shares, and a firm or an oil field is purchased as a whole. Direct and indirect investments in asset holdings are supposed to be perfect substitutes.

[23] In state $g$ households receive transfers: $a_{1t}^g = (p_t(\theta^g) - p_t^f)\theta^g k_t$, $-a_{2t}^g = (p_t^f - p_t(\theta^g))x_t$. The total payment to each country is $l_1 a_{1t}^g = (p_t(\theta^g) - p_t^f)\theta^g K_t = (p_t(\theta^g) - p_t^f)X_t = l_2 a_{2t}^g$. From proposition 6, the price of each contract is $\psi_{t-1}^{ig} = \tilde{\pi}_t^{ig} / R_t$, given that $\theta_{t-1} = \theta^i$, and the net trade of these contracts is $l_1 a_{1t} + l_2 a_{2t} =$ $2\sum_{g=1}^{N} \psi_t^{ig} \theta^g K_t (p_t^f - p_t^*(\theta^g)) = (2K_t / R_t) \sum_{g=1}^{N} \tilde{\pi}_t^{ig} \theta^g (p_t^f - p_t^*(\theta^g)) = 0$ implying the forward oil price as

$$p_t^f = \frac{\sum_{g=1}^{N} \theta^g p_t^*(\theta^g) \tilde{\pi}_t^{ig}}{\sum_{g=1}^{N} \theta^g \tilde{\pi}_t^{ig}} = \frac{\tilde{E}_{t-1}\theta_t p_t^*(\theta_t)}{\tilde{E}_{t-1}\theta_t}.$$

As in the basic model, firms and oil stocks are homogenous, and their numbers are $l_1$, $l_2$, respectively. The capital per firm $k_t$ and the oil stock per field $s_t$ evolve according to the equations

$$k_t = (1-d)k_{t-1} + i_{1t-1}^{kd} + \lambda^{-1} i_{2t-1}^{kd},$$

$$s_t = s_{t-1} - x_{t-1} + (\lambda i_{1t-1}^{sd} + i_{2t-1}^{sd}) / \delta .$$

where $i_{jt-1}^{kd}$, $i_{jt-1}^{sd}$ are the amounts of direct investment in capital and oil stock, respectively, by a country $j = 1{,}2$ household. Labor is immobile and each manufacturing firm in country 1 employs one worker who is a resident of this country. Since the labor supply is inelastic, each employee is rewarded by wage $w_t = f(k_t) - f'(k_t)k_t$.

Let $k_{jt_t}$, $s_{jt_t}$ denote the volumes of capital and oil stock owned by a country $j$ household, and $i_{jt-1}^{ki}$, $i_{jt-1}^{si}$ denote indirect investment by the household in acquisition of a firm's capital or an oil stock, respectively. The household asset holdings evolve as

$$k_{jt} = (1-d)k_{jt-1} + i_{jt-1}^{kd} + i_{jt-1}^{ki},$$

$$s_{jt} = s_{jt-1} - x_{jt-1} + (i_{jt-1}^{sd} + i_{jt-1}^{si}) / \delta .$$

where $x_{jt-1}$ are the volumes of oil extraction related to the oil stock owned by a country $j$ household and satisfying: $x_{1t-1} = \lambda^{-1} x_{t-1}$, $x_{2t-1} = x_{t-1}$ (since $x_{t-1} = X_{t-1} / l_2$, $l_1 + l_2 = 1$). Production units are purchased at market prices $\psi_{t-1}^k$ and $\psi_{t-1}^s$, which are the Tobin's $Q$ for a firm and an oil field, respectively.

The household income on factor holdings in period $t$ is

$y_{jt} = (f'(k_{jt}) - d - \delta\theta_t - \theta_t p(\theta_t))k_{jt} + \delta x_{jt} + r(p_t(\theta)_t s_{jt} + \omega_{jt}$, where $\omega_{jt} = \begin{cases} w_t, & j = 1 \\ 0, & j = 2 \end{cases}$ indicates that the wage is received only by country 1 households. The household budget constraint in period $t-1$ is $\sum_{l=k,s} (i_{jt-1}^{ld} + \psi_{t-1}^{l} i_{jt-1}^{li}) + b_{jt} + c_{jt-1} = y_{jt-1} + R_t b_{jt-1}$ or, after rearranging terms:

$$\begin{aligned} &(\psi_{t-1}^k - 1)i_{jt-1}^{ki} + (\psi_{t-1}^s - 1)i_{jt-1}^{si} + k_{jt} + \hat{s}_{jt} + b_{jt} + c_{jt-1} = \\ &R_t^k k_{jt-1} + R_t^s \hat{s}_{jt-1} + R_t b_{jt-1} + \omega_{jt-1} \end{aligned} \tag{3.13}$$

where $b_{jt}$ is bond holding, $R_t$ is the risk-free bond return.

Households choose at any period consumption $c_{jt}$, the holdings of capital $k_{jt}$, oil stock $\hat{s}_{jt} = \delta s_{jt}$ and bonds $b_{jt}$ under factor prices $\psi_{t-1}^k$, $\psi_{t-1}^s$, wage $w_t$, and the returns $R_t^k$, $R_t^s$, $R_t$. The international financial markets are cleared if $\lambda i_{1t}^{ki} + i_{2t}^{ki} = \lambda i_{1t}^{si} + i_{2t}^{si} =$

$\lambda b_{1t} + b_{2t} = 0$. In equilibrium the volumes of factors per a firm or an oil field are linked to the factor holdings by households as $k_t = k_{1t} + \lambda^{-1} k_{2t}$, $s_t = \lambda s_{1t} + s_{2t}$ (since $k_t = K_t / l_1$, $s_t = S_t / l_2$). The global factor structure is equal to $\kappa_t = \dfrac{\lambda k_{1t} + k_{2t}}{\lambda s_{1t} + s_{2t}} = \dfrac{\lambda k_t}{s_t}$.

*Proposition 7. Under cross-country investment in factors and trade in bonds: i) the Tobin's Q are equal to one for both production factors,* $\psi^k_{t-1} = \psi^s_{t-1} = 1$; *ii) the risk-adjusted expected returns on portfolio and direct investment are equalized across factors and countries,* $j = 1,2$*:*

$$R_t = \widetilde{E}_{jt-1} R^k_t = \widetilde{E}_{jt-1} R^s_t ; \tag{3.14}$$

*iii) For a small oil extraction rate the global factor structure is near optimal:*

$$\kappa_t^2 \approx 2\delta \frac{f'(k_t) - d - \delta \widetilde{E}_{t-1}\theta_t}{\widetilde{E}_{t-1}\theta_t^2}. \tag{3.15}$$

The Tobin's *Q* are equal to one because direct and indirect investments are perfect substitutes for households. According to (3.14), the risk-adjusted expected returns on both factors coincide. The country subscript *j* near the expectation operator in (3.14) means that the risk-adjusted probabilities, generally, differ between the countries since the interstate marginal rates of substitution differ. Nevertheless, the two first moments of the risk-adjusted conditional distribution of $\theta_t$ coincide. From (3.14), the expected factor returns are the same across countries: $\widetilde{E}_{1t-1} R^k_t = \widetilde{E}_{2t-1} R^k_t$ and $\widetilde{E}_{1t-1} R^s_t = \widetilde{E}_{2t-1} R^s_t$ implying coincidence of the first- and second-order moments, respectively: $\widetilde{E}_{1t-1}\theta_t = \widetilde{E}_{2t-1}\theta_t \equiv \widetilde{E}_{t-1}\theta_t$ and $\widetilde{E}_{1t-1}\theta_t^2 = \widetilde{E}_{2t-1}\theta_t^2 \equiv \widetilde{E}_{t-1}\theta_t^2$. Higher-order moments may differ across countries, but they are neglected under the second-order approximation (3.15). As a result, the factor structure given by this equation turns out to be the same as (3.7), implying near optimal investment for a small extraction rate.

In general, the expected factor return is the sum of the risk-free return and the risk premium which is equal to the minus covariance of the marginal utility of consumption and the factor return divided by the mean marginal utility of consumption: $E_{t-1} R^l_t = R_t - Cov_{t-1}(u'_{jt}(c_{jt}), R^l_t) / E_{t-1} u'_{jt}(c_{jt})$ for $l = k, s$ and $j = 1,2$. From (3.14), this covariance is negative for both countries and both factors, and the risk premium is positive. Cross-country investment, thus, results in a positive correlation in the factor returns and in the marginal rates of interstate substitution across countries. This means that households in both countries compose investment risks in a similar way indicating a high degree of international financial integration.

To clarify this issue, consider *a benchmark portfolio* of factor holdings $G_{jt}$ which is built on the weights corresponding to the equilibrium factor structure of the global economy $\kappa_t$: $k_{jt}^G = \alpha(\kappa_t)G_{jt}$, $\hat{s}_{jt}^G = (1-\alpha(\kappa_t))G_{jt}$, where $\alpha(\kappa_t) = \kappa_t/(\delta+\kappa_t)$ is the weight of capital. The return on this portfolio is $R_t^G(\theta_t) = 1+\alpha(\kappa_t)[(f'(k_t) - D(\theta_t) - \phi^{-1}(\theta_t\kappa_t)/\kappa_t]$.[24] The benchmark portfolio provides mutual hedging of countries against fluctuations in the marginal extraction cost. The term $\theta_t p_t(\theta_t)$, which is the source of negative correlation in the factor returns $R_t^k(\theta_t)$ and $R_t^s(\theta_t)$, is diversified away by taking mutually offsetting positions in capital and oil stock holdings.

With the benchmark portfolio, the end-of-period household wealth is represented as $k_{jt} + \hat{s}_{jt} + b_{jt} = G_{jt} + g_{jt} + b_{jt}$, and the beginning-of-period disposable wealth is $R_t^k k_{jt-1} + R_t^s \hat{s}_{jt-1} + R_t b_{jt-1} + \omega_{jt} = R_t^G G_{jt-1} + (R_t^k - R_t^s)g_{jt-1} + R_t b_{jt-1} + \omega_{jt}$, where $g_{jt}$ is *a zero-sum portfolio* of country $j$ consisting of a long/short position in capital and a counterbalancing short/long position in oil stock.. The return on this portfolio is equal to the gap of factor returns $R_t^k - R_t^s$. In equilibrium $\lambda g_{1t} + g_{2t} = 0$, and the zero-sum portfolio provides a gain for one of the countries at the expense of the other one. There is a critical level of $\theta_t$ below which a country with long position in capital gains, while the other country losses. A high degree of financial integration means that the weight of the zero-sum portfolio in household wealth is small or zero. Households in one of the countries have to reject from holding this portfolio if households in the other country do the same, and vice versa. In such equilibrium $g_{jt} = 0$ for $j = 1,2$ and all diversifiable oil price risks are eliminated. All non-diversifiable risks are then contained in the benchmark portfolio $G_{jt}$, and none of the countries benefits at the expense of the other one (under financial autarky, on the contrary, $G_{jt} = 0$ for $j = 1,2$).

*Proposition 8. The return on the benchmark portfolio is represented as* $R_t^G(\theta_t) = R_t + \Delta R_t^G(\theta_t)$, *where* $\Delta R_t^G(\theta_t) = \alpha(\kappa_t)[\delta(\bar{\theta}_t - \theta_t) + (\phi^{-1}(\bar{\theta}_t\kappa_t) - \phi^{-1}(\theta_t\kappa_t))/\kappa_t]$ *is the excess return, and* $\bar{\theta}_t$ *is a threshold oil input coefficient such that* $\Delta R_t^G(\bar{\theta}_t) = 0$.

[24] Since $R_t^k(\theta_t) = 1 + f'(k_t) - D(\theta_t) - p_t(\theta_t)\theta_t$, $R_t^s(\theta_t) = 1 + r(p_t(\theta_t))/\delta$, the benchmark portfolio return is $R_t^G(\theta_t) = R_t^k(\theta_t)\alpha(\kappa_t) + R_t^s(\theta_t)(1-\alpha(\kappa_t)) = 1 + [(f'(k_t) - D(\theta_t) - \phi^{-1}(\theta_t\kappa_t)/\kappa_t]\alpha(\kappa_t)$. The marginal extraction cost term $p_t(\theta_t)\theta_t$ is eliminated from $R_t^G(\theta_t)$ since the static marginal oil rent is $r(p_t) = p_t\xi(p_t) - \phi^{-1}(\xi(p_t))$, and $\xi(p_t) = \theta_t\kappa_t$ under the oil market clearing.

The excess return on the benchmark portfolio $\Delta R_t^G(\theta_t)$ is composed of gains or losses from movements of the oil input coefficient $\theta_t$ relative to the threshold level of this coefficient $\overline{\theta}_t$ ensuring certainty equivalence: $R_t^G(\overline{\theta}_t) = R_t =$ $1 + \alpha(\kappa_t)[f'(k_t) - D(\overline{\theta}_t) - \phi^{-1}(\overline{\theta}_t\kappa_t)/\kappa_t]$. The excess return is obtained from changes in the marginal cost of oil stock maintenance, $\delta(\overline{\theta}_t - \theta_t)$, and in the cost of oil extraction per barrel of oil in the ground, $\phi^{-1}(\overline{\theta}_t\kappa_t) - \phi^{-1}(\theta_t\kappa_t)$, relative to the threshold level.

The formula for excess return is simplified for a special case of Markov chain with a quasi-diagonal transition matrix: $\pi^{ig} > 0$ for $g = i-1, i, i+1$ and $\pi^{ig} = 0$ otherwise, provided that $1 < i < N$, $N \geq 3$. In this case $\theta_t$ can move from one period to the next only to the neighbor states or remain in the same state. If the variation of oil input coefficient $\theta^{i+1} - \theta^i$ is quite small for any state, then the excess return on the benchmark portfolio is: [25]

$$\Delta R_t^G(\theta_t) \approx \alpha(\kappa_t) P_t^f(\overline{\theta}_t)\Delta\overline{\theta}_t,$$

where $\Delta\overline{\theta}_t = \overline{\theta}_t - \theta_t$, $P_t^f(\overline{\theta}_t) = \delta + p_t(\overline{\theta}_t)$, and $\overline{\theta}_t \in (\theta_{i-1}, \theta_{i+1})$. Households in both countries obtain a positive (negative) excess return in period $t$, if the oil input coefficient turns out to be below (above) the threshold level, $\theta_t < (>) \overline{\theta}_t$. The benchmark portfolio rewards or punishes investors for oil demand changes relative to this level measured as $\alpha(\kappa_t)\Delta\overline{\theta}_t$ and priced through the forward oil price $P_t^f(\overline{\theta}_t)$. The latter is based on the threshold oil input coefficient $\overline{\theta}_t$ and the equilibrium factor structure $\kappa_t$, both known in period $t-1$. The higher is this price, the larger is the mean excess return $E_{t-1}\Delta R_t^G(\theta_t)$ providing thereby stronger incentives to invest in both factors.

As a result, the benchmark portfolio of assets ensures that the returns on risky household investment under financial openness are positively correlated. The risks of the benchmark portfolio are defined by the global factor structure and identical for both countries. Holding this portfolio rewards households in both countries for energy-economizing shifts of production technology and encourages risky investment if the oil demand is high.

[25] We have: $\phi^{-1}(\overline{\theta}_t\kappa_t) - \phi^{-1}(\theta_t\kappa_t) \approx z'(\overline{\theta}_t\kappa_t)\Delta\overline{\theta}_t\kappa_t = \varphi(\overline{\theta}_t\kappa_t)\Delta\overline{\theta}_t\kappa_t = p_t(\overline{\theta}_t)\Delta\overline{\theta}_t\kappa_t$. Then $\Delta R_t^G(\theta_t) =$ $\alpha(\kappa_t)[\delta(\overline{\theta}_t - \theta_t) + (\phi^{-1}(\overline{\theta}_t\kappa_t) - \phi^{-1}(\theta_t\kappa_t))/\kappa_t] \approx \alpha(\kappa_t)(\delta + p_t(\overline{\theta}_t))\Delta\overline{\theta}_t = \alpha(\kappa_t)P_t(\overline{\theta}_t)\Delta\overline{\theta}_t$.

## Concluding remarks

The global economy is energy-dependent if energy consumption is comprehensive, capital and energy are to a large extent complementary production factors, and international trade in the energy-careers is highly specialized. The model of this paper reflects these features, though in a stylized form, and emphasizes dynamic interdependence between capital and oil stock. As has been shown, financial openness under energy-dependence is necessary for optimal investment in production factors. Cross-country investment reduces negatively correlated risks and promotes households to build portfolios composed of non-diversifiable and perfectly correlated risks. The benchmark portfolio corresponds to the global factor structure and combines the risks of oil extraction and oil stock maintenance.

The model also captures, in our view, the essence of vertical financial integration between oil consuming and producing economies. If households in both countries have the same investment opportunities, then the cross-country allocation of risks depends on whether or not households hold the zero-sum portfolio. On one hand, absence of this portfolio means a full financial integration in the sense that households diversify away all negatively correlated risks. The distribution of risks between the countries in this case is given by a combination of the benchmark portfolio and the debt holding. This combination is determined, first of all, by cross-country wealth differential and by dependence of household risk attitude on wealth. On the other hand, presence of the zero-sum portfolio in the structure of wealth means that households take a long or short position in the home factor. The short position is possible due to the cross-country heterogeneity of wealth caused by the absence of labor income in country 2 and a high positive correlation of labor and capital incomes in country 1. Alternatively, country 1 may take a long position in oil stock and issue initial debt generating a flow of debt payments to country 2 perfectly correlated with labor income. Analysis of possible portfolio structures is the subject of our further research.

# Appendix

*Proposition 1*. The Euler equation for country 1 is: $u'(c_{1t-1})\dfrac{\partial c_{1t-1}}{\partial k_t} + \beta u'(c_{1t})\dfrac{\partial c_{1t}}{\partial k_t} = 0$.

We have from (1.2), (1.3): $\partial c_{1t-1}/\partial k_t = -1$, and, from (1.16), $\partial c_{1t}/\partial k_t = 1 + f'(k_t) - D - \theta p_t$ implying $u'(c_{1t-1}) = \beta(1 + f'(k_t) - D - \theta p_t)u'(c_{1t})$. Similarly, this equation for country 2 is: $u'(c_{2t-1})\dfrac{\partial c_{2t-1}}{\partial \widehat{s}_t} + \beta u'(c_{2t})\dfrac{\partial c_{2t}}{\partial \widehat{s}_t} = 0$. From (1.2), (1.3) we obtain: $\partial c_{2t-1}/\partial \widehat{s}_t = -1$, and from (1.17), $\partial c_{2t}/\partial \widehat{s}_t = 1 + r(p_t)/\delta$, implying $u'(c_{2t-1}) = \beta(1 + r(p_t)/\delta)u'(c_{2t})$.

*The global planner problem (2.1)-(2.6):* The Lagrangian for this problem is $L^{\mu} = \sum_{t=0}^{\infty}\beta\big(\mu l_1 u(c_{1t}) + l_2 u(c_{2t}) + \eta_t(\theta K_t - X_t)\big)$, where $\eta_t$ is a dual variable related to (2.3). We have, from (2.2), (2.4)-(2.6) that $c_{1t} = (F_t - (P_t\theta K_t - (K_{t+1} - K_t + dK_t) - T_t)/l_1$, $c_{2t} = (P_t X_t - Z_t - \ \delta(S_{t+1} - S_t + X_t) + T_t)/l_2$. The first-order condition for the oil price $P_t$ is $\mu\theta K_t u'(c_{1t}) = X_t u'(c_{2t})$ or $\mu u'(c_{1t}) = u'(c_{2t})$ for (2.3) held as equality. The same first-order condition is obtained for $T_t$.

The first-order condition for oil extraction is $P_t = \partial z/\partial x_t + \delta + \eta_t' \ = p_t + \delta + \eta_t'$, where $\eta_t' = \eta_t / u'(c_{2t})$ is the dual variable in the final good units. The constraint (2.3) holds identically as equality for oil rent-maximizing oil supply $X_t = S_t\xi(p_t)$ implying that $p_t = \varphi(\theta_t K_t / S_t)$ and $\eta_t' = 0$. The optimal oil price is $P_t = p_t + \delta$.

*Proposition 2.* From the capital accumulation equation (2.8), $\partial C_{1t}/\partial K_t =$ $1 + \partial F/\partial K_t - D - \theta p_t - \theta K_t \partial p_t/\partial K_t = R_t^k - \theta K_t \theta\varphi'(\theta\kappa_t)/S_t = R_t^k - \theta^2\varphi'(\theta\kappa_t)\kappa_t$, because $\partial F/\partial K_t = f'(k_t)$ and $p_t = \varphi(\theta\kappa_t)$. From the oil stock accumulation equation (2.9), $\partial C_{2t}/\partial K_t = (\widehat{S}_t/\delta)dr(p_t)/dp_t \cdot \partial p_t/\partial K_t = S_t\xi(p_t)\theta\varphi'(\theta\kappa_t)/S_t = \theta^2\varphi'(\theta\kappa_t)\kappa_t$ because $r'(p_t) = \xi(p_t)$, and $S_t\xi(p_t) = \theta K_t$ from the oil market-clearing condition (1.15).

Similarly, from (2.9), $\partial C_{2t}/\partial \widehat{S}_t = 1 + r(p_t)/\delta \ + (\widehat{S}_t/\delta)dr(p_t)/dp_t \cdot \partial p_t/\partial \widehat{S}_t =$ $R_t^s - S_t\xi(p_t)\theta\varphi'(\theta\kappa_t)K_t/S_t^2\delta = R_t^s - \theta\xi(p_t)\varphi'(\theta\kappa_t)\kappa_t/\delta$, and from (2.8),

$\partial C_{1t} / \partial \widehat{S}_t = -\theta K_t \partial p_t / \partial \widehat{S}_t = \theta K_t \theta \varphi'(\theta \kappa_t) K_t / S_t^2 \delta = \theta \xi(p_t) \varphi'(\theta \kappa_t) \kappa_t / \delta$, because $\theta K_t / S_t = \xi(p_t)$.

*Proposition 3.* The Euler equation for investment in capital is

$$\mu l_1 \left[ u'(c_{1t-1}) \frac{\partial c_{1t-1}}{\partial K_t} + \beta u'(c_{1t}) \frac{\partial c_{1t}}{\partial K_t} \right] + l_2 \left[ u'(c_{2t-1}) \frac{\partial c_{2t-1}}{\partial K_t} + \beta u'(c_{2t}) \frac{\partial c_{2t}}{\partial K_t} \right] = 0 \quad \text{(A2.1)}$$

The capital accumulation equation (2.8) implies that $\partial c_{1t-1} / \partial K_t = l_1^{-1} \partial C_{1t-1} / \partial K_t = -l_1^{-1}$. From (2.10), $\partial c_{1t} / \partial K_t = l_1^{-1} \partial C_{1t} / \partial K_t = (R_t^k - \Delta R_t^k) l_1^{-1}$, $\Delta R_t^k = \theta^2 \varphi'(\theta \kappa_t) \kappa_t$. Similarly, from (2.9) and (2.10) we have that $\partial c_{2t-1} / \partial K_t = 0$, $\partial c_{2t} / \partial K_t = l_2^{-1} \partial C_{2t} / \partial K_t = \Delta R_t^k l_2^{-1}$. Inserting the marginal consumption rates into (A2.1) yields:

$$\mu \left[ -u'(c_{1t-1}) + \beta u'(c_{1t})(R_t^k - \Delta R_t^k \right] + \beta u'(c_{2t}) \Delta R_t^k = 0,$$

or $u'(c_{1t-1}) = \beta R_t^k u'(c_{1t}) - \beta \Delta R_t^k p_t (u'(c_{1t}) - \mu^{-1} u'(c_{2t}))$. Due to (2.7), the second term on the right-hand side of this equation is zero implying $u'(c_{1t-1}) = \beta R_t^k u'(c_{1t})$.

The Euler equation for investment in oil stock is

$$\mu l_1 \left[ u'(c_{1t-1}) \frac{\partial c_{1t-1}}{\partial \widehat{S}_t} + \beta u'(c_{1t}) \frac{\partial c_{1t}}{\partial \widehat{S}_t} \right] + l_2 \left[ u'(c_{2t-1}) \frac{\partial c_{2t-1}}{\partial \widehat{S}_t} + \beta u'(c_{2t}) \frac{\partial c_{2t}}{\partial \widehat{S}_t} \right] = 0. \quad \text{(A2.2)}$$

Equations (2.9) and (2.11) imply that $\partial c_{1t-1} / \partial \widehat{S}_t = 0$, $\partial c_{1t} / \partial \widehat{S}_t = l_1^{-1} \partial C_{1t} / \partial \widehat{S}_t = \Delta R_t^s l_1^{-1}$, where $\Delta R_t^s = \theta \xi(p_t) \varphi'(\theta \kappa_t) \kappa_t / \delta$, and $\partial c_{2t-1} / \partial \widehat{S}_t = l_2^{-1} \partial C_{2t-1} / \partial \widehat{S}_t = -l_2^{-1}$, $\partial c_{2t} / \partial \widehat{S}_t = l_2^{-1} \partial C_{2t} / \partial \widehat{S}_t = (R_t^s - \Delta R_t^s) l_2^{-1}$. Inserting these derivatives into (A2.2) yields:

$$\mu \beta u'(c_{1t}) \Delta R_t^s - u'(c_{2t}) + \beta u'(c_{2t})(R_t^s - \Delta R_t^s) = 0$$

or, rearranging terms, $u'(c_{2t-1}) = \beta R_t^s u'(c_{2t}) + \beta \Delta R_t^s (\mu u'(c_{1t}) - u'(c_{2t}))$. The second term on the right-hand side of this equation is zero and, as a result, (A2.2) is equivalent to $u'(c_{2t-1}) = \beta R_t^s u'(c_{2t})$.

*Proposition 4. (i)* The stochastic Euler equation for capital is

$$\mu l_1 \left[ u'(c_{1t-1}) \frac{\partial c_{1t-1}}{\partial K_t} + \beta E_{t-1} u'(c_{1t}) \frac{\partial c_{1t}}{\partial K_t} \right] + l_2 \left[ u'(c_{2t-1}) \frac{\partial c_{2t-1}}{\partial K_t} + \beta E_{t-1} u'(c_{2t}) \frac{\partial c_{2t}}{\partial K_t} \right] = 0.$$

The marginal consumption rates are defined for any state in the same way as in proposition 2 for the deterministic model. Taking into account the condition of weighted marginal utilities equalization in any state (2.7) yields the Euler equation (2.12) in the same way is in the proof of proposition 3. The stochastic Euler equation for oil stock (2.13) is derived similarly.

*ii)* Condition (3.3) implies that for any $g = 1,...,N$:

$$\frac{u'(c_{1t}(\theta^g))}{\sum_{d=1}^{N} \pi^{id} u'(c_{1t}(\theta^d))} = \frac{u'(c_{2t}(\theta^g))}{\sum_{d=1}^{N} \pi^{id} u'(c_{2t}(\theta^d))},$$

$\theta_{t-1} = \theta^i$, which can be written as $u'(c_{1t}(\theta^g)) / E_{t-1}u'(c_{1t}) = u'(c_{2t}(\theta^g)) / E_{t-1}u'(c_{2t})$ and means equalization of risk-adjusted transitional probabilities: $\tilde{\pi}_{1t}^{ig} = \tilde{\pi}_{2t}^{ig}$ for $i, g = 1,...,N$.

*iii)* The Euler equations (3.4) can be combined as

$$\frac{E_{t-1}R_t^k u'(c_{1t})}{u'(c_{1t-1})} = \frac{E_{t-1}R_t^s u'(c_{2t})}{u'(c_{2t-1})}. \tag{A3.1}$$

From (3.3), the ratios of marginal utilities equalize between countries for $i, g = 1,...,N$:

$$\frac{u'(c_{1t}(\theta^g))}{u'(c_{1t-1}(\theta^i))} = \frac{u'(c_{2t}(\theta^g))}{u'(c_{2t-1}(\theta^i))}.$$

Substituting for these ratios in (A3.1) for each state $g$ yields

$$E_{t-1}R_t^k u'(c_{jt}) = E_{t-1}R_t^s u'(c_{jt}),$$

for $j = 1,2$. Dividing this equation by $E_{t-1}u'(c_{jt})$ yields $\tilde{E}_{t-1}^j R_t^k = \tilde{E}_{t-1}^j R_t^s$ or, since the risk-adjusted conditional expectations coincide for $j = 1,2$, $\tilde{E}_{t-1}R_t^k = \tilde{E}_{t-1}R_t^s$.

*Proposition 5.* The Taylor series expansion of the second order for the left-hand side of (3.6) yields: $\tilde{E}_{t-1}[r(\varphi(\theta_t\kappa_t))/\delta + \theta_t\varphi(\theta_t\kappa_t) + \theta_t\delta] \approx \tilde{E}_{t-1}[r(\varphi(0))/\delta + \theta_t\varphi(0) +$

$r'(0)\varphi'(0)\theta_t\kappa_t/\delta + \theta_t\varphi'(0)\theta_t\kappa_t + \dfrac{d^2r(\varphi(0))}{d(\theta_t\kappa_t)^2}(\theta_t\kappa_t)^2/2\delta + \theta_t\varphi''(0)(\theta_t\kappa_t)^2/2\delta + \theta_t\delta] =$

$\tilde{E}_{t-1}[\varphi'(0)\theta_t^2\kappa_t + \dfrac{d^2r(\varphi(0))}{d(\theta_t\kappa_t)^2}(\theta_t\kappa_t)^2/2\delta + \theta_t\varphi''(0)(\theta_t\kappa_t)^2/2\delta + \theta_t\delta]$, since

$r(0) = \varphi(0) = r'(0) = 0$. We have: $\varphi'(0) = 1/\xi'(0) = 0$, $\varphi''(0) = -\xi''(0)/\xi'(0)^3 = 0$,

$d^2r(\varphi(0))/d(\theta_t\kappa_t)^2 = r''(0)\varphi'(0) + r'(0)\varphi''(0) = r''(0)\varphi'(0) = \xi'(0)/\xi'(0) = 1$. As a result, the right-hand side of (3.6) is approximated as $\tilde{E}_{t-1}[(\theta_t\kappa_t)^2/2\delta + \theta_t\delta] =$

$(\kappa_t^2/2\delta)\tilde{E}_{t-1}\theta_t^2 + \delta\tilde{E}_{t-1}\theta_t$, and (3.6) is represented as

$(\kappa_t^2/2\delta)\tilde{E}_{t-1}\theta_t^2 + \delta\tilde{E}_{t-1}\theta_t = f'(k_t) - d$, implying $\kappa_t^2 = 2\delta\dfrac{f'(k_t) - d - \delta\tilde{E}_{t-1}\theta_t}{\tilde{E}_{t-1}\theta_t^2}$.

*Proposition 6.* The first-order condition for the state-*g* contingent claim is

$$\psi_{t-1}^{ig} u'(c_{jt-1}(\theta^i)) = \beta\pi^{ig} u'(c_{jt}(\theta^g)), \tag{A3.2}$$

for $j = 1,2$, $\theta_{t-1} = \theta^i$, $\theta_t = \theta^g$, and $i, g = 1,...,N$. Summing both sides of (A3.2) over states $g$ implies: $u'(c_{jt-1}) = \beta R_t E_{t-1} u'(c_{jt})$, where $R_t = 1/\sum_{g=1}^{N} \psi_{t-1}^{ig}$. Inserting this into (A3.2) yields: $\psi_{t-1}^{ig} = R_t^{-1} \pi^{ig} u'(c_{jt}(\theta^g))/E_{t-1}u'(c_{jt})$ or $\tilde{\pi}_{jt}^{ig} = \psi_{t-1}^{ig} R_t$, which means equalization of risk-adjusted transition probabilities between the countries: $\tilde{\pi}_{1t}^{ig} = \tilde{\pi}_{2t}^{ig} \equiv \tilde{\pi}_t^{ig}$. Equalization of risk-adjusted expected returns (3.12) is shown in the same way as for proposition 5.

*Proposition 7.* The Euler equations for direct and indirect investment in production factors are $u'(c_{jt-1}) = \beta E_{t-1} R_t^l u'(c_{jt})$, $\psi_{t-1}^l u'(c_{jt-1}) = \beta E_{t-1} R_t^l u'(c_{jt})$, respectively, $l = k, s$, $j = 1,2$. These equations are compatible only if $\psi_{t-1}^l = 1$. The equations for bonds are: $u'(c_{jt-1}) = \beta R_t E_{t-1} u'(c_{jt})$. Combining all these equations yields: $R_t = E_{t-1} R_t^l u'(c_{jt})/E_{t-1}u'(c_{jt})$ or $R_t = \tilde{E}_{jt-1} R_t^k = \tilde{E}_{jt-1} R_t^s$.

The latter equation is similar to (3.6), and yields the similar second-order approximation: $(\kappa_t^2 / 2\delta)\tilde{E}_{jt-1}\theta_t^2 + \delta \tilde{E}_{jt-1}\theta_t \approx f'(k_t) - d$. Since the expected risk-adjusted returns are the same across countries, we have $\tilde{E}_{1t-1} R_t^k = \tilde{E}_{2t-1} R_t^k$, $\tilde{E}_{1t-1} R_t^s = \tilde{E}_{2t-1} R_t^s$ implying $\tilde{E}_{1t-1}\theta_t = \tilde{E}_{2t-1}\theta_t \equiv \tilde{E}_{t-1}\theta_t$ and $\tilde{E}_{1t-1}\theta_t^2 = \tilde{E}_{2t-1}\theta_t^2 \equiv \tilde{E}_{t-1}\theta_t^2$, respectively. The first- and second-order risk-adjusted moments of $\theta_t$ coincide, and we obtain (3.15).

*Proposition 8*: proof is straightforward.

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